

# NC Automorphisms of nc-bounded domains

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## Abstract

We establish rigidity (or uniqueness) theorems for nc automorphisms which are natural extensions of classical results of H. Cartan and are improvements of recent results. We apply our results to nc-domains consisting of unit balls of rectangular matrices.

## 1 Introduction

Holomorphic automorphisms of domains in  $\mathbb{C}^d$  have been studied since the work of H. and E. Cartan in the 1930's [7], [6]. A holomorphic function can be thought of as a generalized polynomial, and they can be evaluated not just on tuples of complex numbers, but also on tuples of commuting matrices or commuting operators whose spectrum is in the domain of the function [20]. An nc-function (nc stands for non-commutative) is a generalization of a free polynomial, (*i.e.* a polynomial in non-commuting variables), and it is natural to evaluate them on tuples of matrices or operators.

To describe nc-functions (following [13] for instance), we must first establish some notation. Let  $\mathbb{M}_n$  denote the  $n$ -by- $n$  complex matrices, and  $\mathbb{M}_n^d$  the  $d$ -tuples of  $n$ -by- $n$  matrices. We shall let  $\mathbb{M}^{[d]}$  denote the disjoint union  $\cup_{n=1}^{\infty} \mathbb{M}_n^d$ . Given  $x = (x^1, \dots, x^d)$  in  $\mathbb{M}_n^d$  and  $y = (y^1, \dots, y^d)$  in  $\mathbb{M}_m^d$ , by  $x \oplus y$  we mean the element  $(x^1 \oplus y^1, \dots, x^d \oplus y^d)$  of  $\mathbb{M}_{m+n}^d$ . If  $x \in \mathbb{M}_n^d$  and  $s, t \in \mathbb{M}_n$ , by  $sxt$  we mean  $(sx^1t, \dots, sx^dt)$ .

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A free polynomial  $p$  in  $d$  variables can be thought of as a function defined on  $\mathbb{M}^{[d]}$ , and as such it has the following properties:

- (i) If  $x$  is in  $\mathbb{M}_n^d$ , then  $p(x) \in \mathbb{M}_n$ .
- (ii) If  $x$  and  $y$  are in  $\mathbb{M}^{[d]}$ , then  $p(x \oplus y) = p(x) \oplus p(y)$ .
- (iii) If  $x \in \mathbb{M}_n^d$  and  $s \in \mathbb{M}_n$  is invertible, then  $p(s^{-1}xs) = s^{-1}p(x)s$ .

**Definition 1.** An nc-set is a set  $\Omega \subseteq \mathbb{M}^{[d]}$  such that  $\Omega_n := \Omega \cap \mathbb{M}_n^d$  is an open set for each  $n$ , and such that  $\Omega$  is closed with respect to direct sums and joint unitary equivalence (*i.e.* for all  $x \in \Omega_n$  and for all  $u$  unitary in  $\mathbb{M}_n$ , we have  $u^{-1}xu \in \Omega$ ). If an nc-set  $\Omega$  has the property that  $\Omega_n$  is connected for every  $n$ , we shall call it an nc-domain.

An nc-function is a function on an nc-set that mimics the properties (i) - (iii) above of free polynomials.

**Definition 2.** An nc-function  $f$  on an nc-set  $\Omega$  is a function with the following three properties:

- (i) If  $x$  is in  $\Omega_n$ , then  $f(x) \in \mathbb{M}_n$  (we say  $f$  is graded if this occurs).
- (ii) If  $x$  and  $y$  are in  $\Omega$ , then  $f(x \oplus y) = f(x) \oplus f(y)$ .
- (iii) If  $x \in \Omega_n$ ,  $s \in \mathbb{M}_n$  is invertible, and  $s^{-1}xs \in \Omega$ , then  $f(s^{-1}xs) = s^{-1}f(x)s$ .

An nc-map  $\Phi$  on an nc-domain  $\Omega \subseteq \mathbb{M}^{[d]}$  is a  $d$ -tuple of nc-functions. If  $\Phi$  is an nc-map on  $\Omega$  that is also a bijection onto  $\Omega$ , we call it an nc-automorphism.

Bounded symmetric domains in  $\mathbb{C}^d$  have been characterized by E. Cartan [6], and in the course of the proof automorphisms of such domains were described.

We are interested in the following questions about nc-automorphisms.

*Question 1. (Rigidity)* If  $\Phi$  is an nc-automorphism of  $\Omega$ , is it uniquely determined by its action on  $\Omega \cap \mathbb{M}_n^d$  for some fixed  $n$ ?

*Question 2. (Extendibility)* If  $F : \Omega \cap \mathbb{M}_n^d \rightarrow \Omega \cap \mathbb{M}_n^d$  is a biholomorphic map which respects similarities, is there an nc-automorphism  $\Phi : \Omega \rightarrow \Omega$  such that  $\Phi|_{\Omega \cap \mathbb{M}_n^d} = F$ ?

*Question 3.* What groups can arise as the automorphism group of an nc-domain?

A set  $\Omega \subset \mathbb{M}^{[d]}$  is called *nc-bounded* if for each  $n$ , there exists a constant  $M_n$  such that

$$\forall z \in \Omega_n, \quad \|z\| < M_n.$$

In Theorem 7 we show that one always has rigidity on nc-bounded domains that contain the origin. For such domains, the possible automorphism groups are therefore no more than certain subgroups of the automorphism groups of bounded domains (and we substantially answer questions 1, 2 and 3 for circular bounded domains that contain the origin). In Example 15, we show that many different domains can have the same automorphism groups.

In Section 4, we consider the Cartan domain of type I, the set  $R_{pq}$  of  $p$ -by- $q$  contractive matrices. The obvious nc-domain containing this, where numbers are replaced by  $n$ -by- $n$  matrices, we call  $\mathbf{R}_{pq}$ . In Theorem 13, we show that when  $p \neq q$ , all automorphisms of  $R_{pq}$  extend; but when  $p = q$ , only those automorphisms that do not involve the transpose extend.

In this paper, we restrict our attention to nc-bounded domains, as the unbounded case is much more complicated (see *e.g.* Example 9).

This note continues work of Popescu in [16, 17, 18] and of Helton, Klep, McCullough and Slingend, in [12].

## 2 Background on nc-functions

The recent monograph [13] by D. Kaliuzhnyi-Verbovetskyi & V. Vinnikov gives an introduction to nc-functions. Unless an additional hypothesis of continuity (or boundedness) is added, nc-functions can behave badly.

*Example 3.* Let  $d = 1$ , and define a function  $f$  on Jordan blocks by sending a Jordan block with 0 eigenvalues to the zero matrix of the same size, and a Jordan block with non-zero eigenvalues to the identity matrix of that size. Extend  $f$  by direct sums to any matrix in Jordan canonical form, and then by similarity to any matrix. The function  $f$  is then an nc-function which is manifestly discontinuous.

Let  $\sigma$  denote the disjoint union topology on  $\mathbb{M}^{[d]}$ : a set  $U$  is in  $\sigma$  if and only if  $U \cap \mathbb{M}_n^d$  is open for every  $n$ . (This topology is called the finitely open topology in [13]).

It was proved in [1] that if an nc-function  $f$  on an nc-set  $\Omega$  is  $\sigma$  locally bounded, in the sense that

$$\forall z \in \Omega, \exists U \in \sigma \text{ s.t. } z \in U \text{ and } f|_{\Omega \cap U} \text{ is bounded,}$$

then  $f : \Omega \rightarrow \mathbb{M}^{[1]}$  is  $\sigma$ - $\sigma$  continuous, and in [10], it was shown that this in

turn implied that  $f$  was holomorphic, in the sense that

$$\forall n \in \mathbb{N}, \forall z \in \Omega_n, \forall h \in \mathbb{M}_n^d, \quad \exists Df(z)[h] = \lim_{t \rightarrow 0} \frac{1}{t} [f(z + th) - f(z)]. \quad (2.1)$$

Putting these results together, we conclude

**Proposition 4.** *An nc-map  $f$  into an nc-bounded domain is automatically  $\sigma$ - $\sigma$  continuous, and holomorphic in the sense of (2.1).*

### 3 Rigidity

A domain is called *circular* if it is invariant under multiplication by unimodular scalars.

The following lemmas are classical and due to H. Cartan.

**Lemma 5** (H. Cartan ([8, Théorème VII, p. 30])). *Let  $\mathcal{D} \subseteq \mathbb{C}^d$  be a bounded domain,  $z_0 \in \mathcal{D}$  and  $\phi: \mathcal{D} \rightarrow \mathcal{D}$  a biholomorphic automorphism with  $\phi(z_0) = z_0$  and  $\phi'(z_0) = I_n$ . Then  $\phi$  is the identity.*

**Lemma 6** ([7], [8, Théorème VI]). *If  $\mathcal{D}$  is a bounded circular domain in  $\mathbb{C}^d$  containing 0, and  $F: \mathcal{D} \rightarrow \mathcal{D}$  is a biholomorphic automorphism of  $\mathcal{D}$  with  $F(0) = 0$ , then  $F$  is the restriction to  $\mathcal{D}$  of an invertible linear map.*

**Theorem 7.** *Let  $\Omega$  be an nc-domain that is nc-bounded. Let  $\Phi = (\Phi^1, \dots, \Phi^d)$  be an nc-automorphism of  $\Omega$ . Suppose that for some  $m \in \mathbb{N}$ , we have  $0 \in \Omega \cap \mathbb{M}_m^d$  and  $(\Phi|_{\Omega_m})(0) = 0$ .*

(i) *If in addition  $(\Phi|_{\Omega_m})'(0)$  is the identity, then  $\Phi$  is the identity on all of  $\Omega$ .*

(ii) *If instead we suppose also that  $\Omega$  is a circular nc-domain, then there is an invertible linear map  $F$  on  $\mathbb{C}^d$  such that  $\Phi(Z) = (F \otimes \text{id}_n)(Z)$  for  $Z \in \Omega_n$  (by which we mean that each  $d$ -tuple  $(\Phi(Z))_{i,j}$  formed from the  $(i, j)$  coordinates of the  $d$ -tuple of  $n \times n$  matrices  $\Phi(Z)$  is given by  $F(Z_{i,j})$  where  $Z = (Z^1, \dots, Z^d)$ , and  $Z_{i,j} = (Z_{i,j}^1, \dots, Z_{i,j}^d)$  again denotes the  $(i, j)$  coordinates).*

*Proof.* (i) By Lemma 5,  $\Phi|_{\Omega_m}$  is the identity.

As  $\Omega_m = \Omega \cap \mathbb{M}_m^d$  is open, there is some  $\varepsilon > 0$  such that if  $z = (z^1, \dots, z^d) \in \mathbb{M}_m^d$  has each  $\|z^j\| < \varepsilon$ , then  $z \in \Omega_m$ .

Now, fix a positive integer  $k$ . For  $Z^j \in \mathbb{M}_{km}$ , we write  $\hat{Z}^j$  for the  $d$ -tuple  $(Z^1, \dots, Z^d) \in \mathbb{M}_{km}^d$  that has  $Z^i = 0$  when  $i \neq j$ , and whose  $j^{\text{th}}$  entry is  $Z^j$ . If  $Z^j$  is the direct sum of  $k$  matrices from  $\mathbb{M}_m$ , and if  $\|Z^j\| \leq \varepsilon$ , then  $\Phi(\hat{Z}^j) = \hat{Z}^j$ , by the direct sum property of nc-maps. As this applies to  $\zeta Z^j$  for  $|\zeta| < 1$  ( $\zeta \in \mathbb{C}$ ), it follows that the directional derivative  $((\Phi|_{\Omega_{km}})'(0))(\hat{Z}^j) = \hat{Z}^j$ . By linearity of the Fréchet derivative  $(\Phi|_{\Omega_{km}})'(0)$  we may drop the restriction that  $\|Z^j\| < \varepsilon$ . In particular the conclusion holds when  $Z^j \in \mathbb{M}_{km}$  is a diagonal matrix.

By the chain rule and similarity invariance

$$(\Phi|_{\Omega_{km}})(s^{-1}Zs) = s^{-1}(\Phi|_{\Omega_{km}})(Z)s$$

of the map  $\Phi|_{\Omega_{km}}$  (valid for all sufficiently small  $Z$  once  $s$  is fixed), we must have that  $(\Phi|_{\Omega_{km}})'(0)$  has the invariance property

$$((\Phi|_{\Omega_{km}})'(0))(s^{-1}Zs) = s^{-1}(((\Phi|_{\Omega_{km}})'(0))(Z))s \quad (3.1)$$

Choosing  $Z^j$  diagonalisable and  $s$  such that  $s^{-1}Z^j s$  is diagonal (3.1) yields  $s^{-1}(((\Phi|_{\Omega_{km}})'(0))(\hat{Z}^j))s = s^{-1}\hat{Z}^j s$  and hence  $((\Phi|_{\Omega_{km}})'(0))(\hat{Z}^j) = \hat{Z}^j$  provided  $Z^j$  is diagonalisable. By density of the diagonalisable matrices we can then make the same conclusion with  $Z^j$  arbitrary and then linearity of  $(\Phi|_{\Omega_{km}})'(0)$  forces it to be the identity.

By Lemma 5 again,  $\Phi|_{\Omega_{km}}$  must be the identity.

Now choose some  $n$ , not necessarily a multiple of  $m$ , such that  $\Omega_n$  is non-empty. Let  $Z \in \Omega_n$ . The direct sum of  $m$  copies of  $Z$  is in  $\Omega_{mn}$ , and  $\Phi(\oplus_{i=1}^m Z) = \oplus_{i=1}^m Z$  by the first part of the proof. As  $\Phi$  preserves direct sums, this means that  $\Phi(Z) = Z$ .

- (ii) By Lemma 6, we know  $\Phi|_{\Omega_m}$  is linear, If  $m = 1$  we take  $F$  to be  $\Phi|_{\Omega_m}$ . For  $m > 1$  we need a brief argument to find  $F$ .

Similarity invariance

$$(\Phi|_{\Omega_m})(s^{-1}Zs) = s^{-1}(\Phi|_{\Omega_m})(Z)s$$

(guaranteed by Definition 2 to hold for  $Z, s^{-1}Zs \in \Omega_m$ ) must hold globally for  $Z \in \mathbb{M}_m^d$  in view of linearity. Let  $E_{i,k}$  denote the standard matrix units in  $\mathbb{M}_m$  and choose  $Z = (z^1 E_{1,1}, \dots, z^d E_{1,1})$  (so that  $Z$  is supported on the  $(1,1)$  entries). Consider a block diagonal  $s = 1 \oplus t$

with 1 in the  $(1, 1)$  entry but arbitrary invertible  $(m - 1) \times (m - 1)$  block  $t$ . Since the matrices that commute with all such  $1 \oplus t$  are those of the form  $F \oplus \alpha I_{m-1}$  (for scalars  $F$  and  $\alpha$ ) we see that  $\Phi|_{\Omega_m}(Z)$  must take the form

$$(F^1(z) \oplus \alpha^1(z) I_{m-1}, \dots, F^d(z) \oplus \alpha^d(z) I_{m-1}), \text{ with } z = (z^1, \dots, z^d),$$

for some scalar-valued linear  $F^1, \dots, F^d, \alpha^1, \dots, \alpha^d: \mathbb{C}^d \rightarrow \mathbb{C}$ .

However, notice that (by Lemma 6) we also know that  $\Phi|_{\Omega_{2m}}$  is linear, and by the direct sum property  $\Phi(Z \oplus 0) = \Phi(Z) \oplus 0$ . The similarity argument applied to  $\mathbb{M}_{2m}^d$  forces  $\alpha^j = 0$  ( $1 \leq j \leq d$ ). Thus

$$\Phi|_{\Omega_m}(z^1 E_{1,1}, \dots, z^d E_{1,1}) = (F^1(z) E_{1,1}, \dots, F^d(z) E_{1,1})$$

(with  $z = (z^1, \dots, z^d)$ ). Using similarity with  $s$  a transposition allows us to conclude that the same must hold for  $E_{k,k}$  replacing  $E_{1,1}$ . Taking  $s = I_n - E_{i,k}$  for  $i \neq k$  we have  $s^{-1} E_{ii} s = E_{ii} - E_{ik}$ , and together with linearity we deduce the relation with  $E_{k,k}$  replaced by  $E_{i,k}$ . Clearly  $F = (F^1, \dots, F^d): \mathbb{C}^d \rightarrow \mathbb{C}^d$  must be invertible (since  $\Phi$  is) and we have the desired conclusion on  $\Omega_m$ .

By Lemma 6, we know  $\Phi|_{\Omega_{km}}$  is linear for each  $k \in \mathbb{N}$ . For  $Z^j \in \mathbb{M}_{km}$ , if  $Z^j$  is diagonal we must have  $\Phi(\hat{Z}^j)$  of the required form. The similarity property and density of the diagonalisable matrices in  $\mathbb{M}_{km}$  allows us to extend to arbitrary nonzero  $Z^j$ . Then by linearity this extends to arbitrary  $d$ -tuples  $Z = (Z^1, \dots, Z^d) \in \mathbb{M}_{km}^d$ .

Finally if  $\Omega_n$  is nonempty for some  $n$  we can apply the result just obtained for  $\Omega_{nm}$  together with the direct sum property for  $Z \oplus 0 \oplus \dots \oplus 0$  (where  $Z \in \mathbb{M}_n$  and we have  $(m - 1)$  zero summands) to establish the desired conclusion for  $\Phi|_{\Omega_n}$ .  $\square$

Popescu's Cartan uniqueness results [17, §1] can be viewed as similar in spirit for the case of special domains (row contractions) to Theorem 7. In the case  $m = 1$ , the result of [10, Corollary 4.1 (2)] is part (i) while [11, Theorem 21] implies (ii).

*Example 8.* The matrix polydisk. This is the set

$$\mathbf{D} = \{x \in \mathbb{M}^{[d]} : \|x^j\| < 1, 1 \leq j \leq d\}.$$

The set of automorphisms of  $\mathbf{D}$  is the set

$$\{\Phi(x) = \sigma \circ (m^1(x^1), \dots, m^d(x^d)) : \sigma \in \mathbf{S}_d, m^j \in \text{Aut}(\mathbb{D})\}. \quad (3.2)$$

Here  $\mathbf{S}_d$  is the symmetric group on  $d$  variables, and  $\text{Aut}(\mathbb{D})$  is the Möbius group of automorphisms of the disk. Each Möbius transformation of the form

$$m(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}$$

extends to matrices in the obvious way:

$$m(Z) = e^{i\theta} (Z - aI)(1 - \bar{a}Z)^{-1}.$$

Indeed, by von Neumann's inequality, every  $\Phi$  in (3.2) extends to an automorphism of  $\mathbf{D}$ . That this comprises everything follows from observing that all automorphisms of the polydisk  $\mathbb{D}^d = \mathbf{D} \cap \mathbb{M}_1^d$  are of this form, and so by Theorem 7 they have a unique extension to higher levels. As they are invertible, they must be automorphisms.  $\triangleleft$

*Example 9.* Theorem 7 fails if boundedness is dropped. Consider, for example, the nc-set

$$\Omega = \{(x, y, z) \in \mathbb{M}^{[3]} : \|xy - yx\| < 1\}.$$

Let

$$\Phi(x, y, z) = (x, y, z + h(xy - yx)),$$

where  $h : \mathbb{C} \rightarrow \mathbb{C}$  is any non-constant entire function mapping 0 to 0. Then  $\Phi$  is an automorphism, and  $\Phi|_{\Omega \cap \mathbb{M}_1^3}$  is the identity, but  $\Phi$  is not the identity on level 2.  $\triangleleft$

## 4 Extendibility in $\mathbf{R}_{pq}$

Let  $R_{pq}$  denote the  $p$ -by- $q$  matrices of norm less than 1. We shall extend this to an nc-domain in  $\mathbb{M}^{[d]}$ , where  $d = pq$ , by

$$\mathbf{R}_{pq} := \bigcup_{n=1}^{\infty} \{(x^1, x^2, \dots, x^d) \in \mathbb{M}_n^d : \left\| \begin{pmatrix} x^1 & \dots & x^q \\ x^{q+1} & \dots & x^{2q} \\ \vdots & & \\ x^{(p-1)q+1} & \dots & x^{pq} \end{pmatrix} \right\| < 1\}.$$

Let  $\gamma : \mathbb{M}_n^d \rightarrow \mathbb{M}_{pq}^d$  be the map that takes  $d$  matrices and makes them into a block  $p$ -by- $q$  matrix, filling in left to right and then top to bottom. Then  $\mathbf{R}_{pq} = \{x : \|\gamma(x)\| < 1\}$ . When we speak of an nc automorphism of  $\mathbf{R}_{pq}$ , strictly speaking we mean an nc automorphism of  $\gamma^{-1}(\mathbf{R}_{pq})$ .

In the special case  $q = 1$ ,  $R_{p1}$  is just the unit ball in  $\mathbb{C}^p$ , and its automorphisms are well-known (see *e.g.* [19, Thm. 2.2.5]). The set  $\mathbf{R}_{1q}$ , the row-contractions, was studied by G. Popescu in [15, 16] and [17].

The automorphisms of  $R_{pq}$  are given by a similar formula to the case of the ball. L. Harris showed [9] that they are of the following form.

**Theorem 10.** *[Harris] Every holomorphic automorphism of  $R_{pq}$  is of the form  $LH_A$  where  $L$  is a linear isometric automorphism of  $R_{pq}$ ,  $A$  is an element of  $R_{pq}$ , and*

$$H_A(x) = (I_p - AA^*)^{-1/2}(x + A)(I_q + A^*x)^{-1}(I_q - A^*A)^{1/2}.$$

First, let us consider that automorphisms that map 0 to 0, which are the linear ones. K. Morita [14] classified the linear isometries of  $R_{pq}$ , and the square case differs from the rectangular case, because the transpose is an isometry.

**Theorem 11.** *[Morita] If  $p \neq q$ , all linear automorphisms of  $R_{pq}$  are of the form  $x \mapsto UxV$ , where  $U$  is a  $p$ -by- $p$  unitary and  $V$  is a  $q$ -by- $q$  unitary. If  $p = q$ , the set of linear automorphisms consists of  $x \mapsto UxV$  and  $x \mapsto Ux^tV$ .*

The map  $x \mapsto UxV$  extends to the nc automorphism of  $\mathbf{R}_{pq}$  given by  $Z \mapsto (\text{id} \otimes U)Z(\text{id} \otimes V)$ ; but the transpose does not extend.

**Lemma 12.** *If  $p > 1$ , the map  $x \mapsto x^t$  does not extend to an nc automorphism of  $\mathbf{R}_{pp}$ .*

*Proof.* Using Theorem 7, if the transpose did extend, the extension would map  $(X_{i,j}) \in \mathbb{M}_p(\mathbb{M}_n(\mathbb{C}))$  to  $(X_{j,i})$ , and so this reduces to the well-known fact that the transpose map is not a complete isometry of  $\mathbb{M}_p$ .  $\square$

Let us turn now to  $H_A$ . The map  $H_A$  extends to an nc map from  $\mathbf{R}_{pq}$  to  $\mathbf{R}_{pq}$  given by

$$H_A(Z) = (I_{n,p} - \text{id} \otimes AA^*)^{-1/2}(Z + \text{id} \otimes A)(I_{n,q} + (\text{id} \otimes A^*)Z)^{-1}(I_{n,q} - \text{id} \otimes A^*A)^{1/2} \quad (4.1)$$

Here,  $\text{id}$  means  $\text{id}_{\mathbb{C}^n}$ , and  $I_{n,r}$  denotes  $\text{id}_{\mathbb{C}^n \otimes \mathbb{C}^r}$ .



A calculation shows that

$$\begin{aligned} \text{id} - H_A(W)^* H_A(Z) &= (I_{n,q} - \text{id} \otimes A^* A)^{1/2} (I_{n,q} + W^* (\text{id} \otimes A))^{-1} \\ &\quad (I_{n,q} - W^* Z) (I_{n,q} + (\text{id} \otimes A^*) Z)^{-1} (I_{n,q} - \text{id} \otimes A^* A)^{1/2}. \end{aligned} \quad (4.2)$$

Letting  $W = Z$  proves that  $H_A$  maps  $\mathbf{R}_{pq}$  to  $\mathbf{R}_{pq}$ , and as  $H_{-A}$  is the inverse of  $H_A$ , it must be an automorphism.

Putting these results together, we get the following theorem. The case  $p = 1$  was proved by Popescu [17, Thm. 1.5]. The general case was proved by Helton, Klep, McCullough and Slinglend [12, Thm. 1.7], though their hypotheses are stronger. The linear case was proved by D. Blecher and D. Hay [5].

**Theorem 13.** *If  $p \neq q$ , then every automorphism of  $R_{pq}$  extends uniquely to an automorphism of  $\mathbf{R}_{pq}$ . If  $p = q$ , the automorphisms of the form  $x \mapsto UH_A(x)V$  extend uniquely to  $\mathbf{R}_{pp}$ , and the automorphisms of the form  $x \mapsto UH_A(x)^t V$  (when  $p > 1$ ) do not extend to automorphisms of  $\mathbf{R}_{pp}$ .*

By a result of J. Ball and V. Bolotnikov [4], (see also [3] and [2])  $H_A$  extends to an endomorphism of the *commuting* elements of  $\gamma^{-1}(\mathbf{R}_{pq})$  if and only if there is some function  $F$  so that

$$I - H_A(w)^* H_A(z) = F(w)^* (I - w^* z) F(z)$$

as a kernel on  $R_{pq}$ . This is true, as (4.2) shows. So the nc automorphisms of  $\{x \in \gamma^{-1}(\mathbf{R}_{pq}) : x^i x^j = x^j x^i, \forall 1 \leq i, j \leq d\}$  are the same as the nc automorphisms of  $\gamma^{-1}(\mathbf{R}_{pq})$ . This phenomenon has also been explored in [1].

*Question 4.* The automorphisms of  $\mathbf{R}_{pq}$  are not transitive at any level  $n \geq 2$ . What can one say about the orbits?

Theorem 13 can be extended slightly. For  $S$  a subset of  $\mathbb{N}$  that is closed under addition, let  $\mathbf{R}_{pq}(S)$  be defined by  $\mathbf{R}_{pq}(S) \cap \mathbb{M}_n^d$  is  $\mathbf{R}_{pq} \cap \mathbb{M}_n^d$  if  $n \in S$ , and empty otherwise.

**Proposition 14.** *Let  $S$  be any non-empty sub-semigroup of  $\mathbb{N}$ . Then the automorphisms of  $\mathbf{R}_{pq}(S)$  are the same as the automorphisms of  $\mathbf{R}_{pq}$ , and are uniquely determined by their action on any non-empty level.*

*Example 15.* Extendibility can fail if the pieces of  $\Omega$  at different levels are not somehow alike. For example, let  $d = 1$ , and  $R > 1$ . Define  $\Omega$  by  $\Omega \cap \mathbb{M}_1 = \mathbb{D}$ ,

and  $\Omega \cap \mathbb{M}_n = \{x : \|x\| < R\}$ . The automorphisms of  $\Omega \cap \mathbb{M}_1$  are the Möbius maps, but only multiplication by  $e^{i\theta}$  extends to be an automorphism of  $\Omega$ .

But there are many other choices of nc-domain  $\Omega \supset \mathbf{R}_{11}$  that have the same automorphism group. For example, let  $r_1 = 1$ , and let  $(r_n)_{n=1}^\infty$  be any non-decreasing sequence. Define  $\Omega$  by

$$\Omega_n = \{x \in \mathbb{M}_n : \exists s \in \mathbb{M}_n, \text{ with } \|s\| \|s^{-1}\| \leq r_n, \text{ and } \|s^{-1}xs\| < 1\}. \quad (4.3)$$

Then  $\Omega$  is an nc-domain, and its automorphism group is the set of Möbius maps.  $\triangleleft$

*Question 5.* Let  $U \subset \mathbb{C}^d$  be a bounded symmetric domain. Is there an nc-domain  $\Omega \subset \mathbb{M}^{[d]}$  such that  $\Omega_1 = U$  and such that the automorphism group of  $\Omega$  equals the automorphism group of  $U$ ?

## References

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